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SECONDARY PERTURBATIONS OF BENJAMIN WAVES†

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The evolution of small two-dimensional perturbations of boundary flow in the region preceding the laminar-turbulent transition zone is investigated using a four-deck asymptotic model. It is assumed that the main flow is a two-dimensional boundary layer flow distorted by a wave of finite amplitude that is periodic in time and with a longitudinal coordinate. The problem is reduced to investigating the solutions of a linearized Benjamin–Ono equation, which describes secondary perturbations of a periodic Benjamin wave. The explicit expressions obtained for the structure of the perturbations confirm the neutral stability of the main flow with respect to two-dimensional perturbations. An exact solution of the problem of the passage of a Tollmien–Schlichting wave through a region perturbed by a solitary Benjamin wave is constructed by analysing the stability of the periodic.

1. INTRODUCTION

According to the Landau–Hopf hypothesis [1], the laminar-turbulent transition in viscous flows represents a sequence of bifurcations, each of which is a consequence of the loss of stability of the intermediate state of the flow, corresponding to a certain periodic fluid motion. The transition of subharmonic and Klebanoff modes [2] is also developed precisely in accordance with this scenario: initially the steady flow in the boundary layer loses stability with respect to the Tollmien–Schlichting mode, and then, on attaining pulsations of a certain threshold amplitude, a new destabilization of the resultant periodic motion occurs, and an increase in the secondary perturbations lead to rapid breakdown of the laminar flow.

A detailed analysis of these processes is extremely complex, and hence for a theoretical investigation of the laminar-turbulent transition, direct numerical simulation is usually employed, namely, a calculation of the solution of a mixed boundary-value problem for the complete system of Navier–Stokes equations using finite difference methods. The results obtained must be treated in the same way as the results of an experimental investigation.

In this connection asymptotic models, which approximately described unsteady fluid motions in the pretransition region (where the Reynolds number is fairly high), have acquired a special role. Their use enables one to simplify the system of equations considerably, while retaining a qualitatively correct description of the main characteristics of the wave motion in the transition region (the dispersion law, the non-linearity, etc.)

We will consider subsonic two-dimensional laminar boundary layer flow of a viscous gas. We will introduce the following notation for the local values of the dimensional parameters of the flow: δ° is the characteristic thickness of the boundary layer and u° is the flow velocity on its outer boundary. The Reynolds number $R = u^\circ\delta^\circ/\nu^\circ$ will be assumed to be infinitely large (ν° is the characteristic value of the kinematic viscosity). After the primary loss of stability of the steady flow, the perturbation experiences parametric amplification along the path corresponding to the neighbourhood of the lower branch of the neutral curve; during evolution of the perturbation the point corresponding to the wave parameters is moved away from the lower branch, being displaced into the region of lower wavelengths and higher Reynolds numbers. In this region the characteristic wavelength λ° satisfies the inequality

$$1 \ll \lambda^\circ/\delta^\circ \ll R^{1/4}$$

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(the right-hand boundary of the region considered corresponds to the neighbourhood of the lower branch of the neutral stability curve [3]), while the characteristic time scale τ° of the unsteady perturbations is defined by the relation $\tau^\circ u^\circ/\delta^\circ = (\lambda^\circ/\delta^\circ)^2$.

It has been shown [4], that perturbations, characterized by such time and length scales, have a four-deck structure. In the principal approximation they are accompanied by local displacements of the velocity and density profiles in a direction normal to the surface by an amount of the order of $\Delta^\circ = \delta^{\circ 2}/\lambda^\circ$.

Using the scales τ°, α°, Δ°, we will introduce dimensionless independent variables t (the time) and x (the streamwise coordinate), and also a dimensionless quantity A, characterizing the local displacement thickness, taken with the opposite sign. In the case of two-dimensional perturbations [4], the function $A(t, x)$ satisfies the Benjamin–Ono integrodifferential equation

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\partial^2 A(t,\xi)/\partial\xi^2}{\xi - x}d\xi \tag{1.1}$$

first obtained in [5] when describing the non-linear evolution of internal waves in a stratified liquid of infinite depth. The improper integral on the right-hand side of this equation is understood in the sense of the principal value.

In the case of small oscillations

$$A = \varepsilon\exp(ikx - i\omega t) + \text{c.c.} + O(\varepsilon^2)$$

Eq. (1.1) gives the dispersion relation

$$\omega = k|k| \tag{1.2}$$

which approximates quite well the dependence of the frequency ω of the Tollmien–Schlichting wave on the wave number k, calculated from the classical theory of the stability of parallel flows for values of R corresponding to the onset of laminar-turbulent transition (see [6]). Moreover, an accurate periodic solution of non-linear equation (1.1) (the Benjamin wave [5]) perfectly describes the form of pulsations observed experimentally, introduced into the flow by a Tollmien–Schlichting wave of finite amplitude in the pretransition part of the boundary layer [7]. These facts justify the use of a four-deck asymptotic scheme [4, 8] to describe the non-linear processes in the laminar-turbulent transition zone.

In this paper we investigate the structure of small two-dimensional perturbations, which propagate on a background of waves of finite amplitude [5]. The results of the investigation were briefly described previously in [9].

2. SECONDARY PERTURBATIONS OF THE PERIODIC WAVE

We will first consider the problem of two-dimensional perturbations of a periodic Benjamin wave. We will seek a solution of Eq. (1.1) in the form of the expansion

$$A = A_0 + \varepsilon A_1 + O(\varepsilon^2)$$

Equation (1.1) is invariant under the replacement $x - ct \to x, A - c \to A$. In this connection, we will investigate the perturbed flow in a reference frame moving with the velocity of the fundamental wave. In this case the function $A_0(x)$ satisfies the stationary analogue of Eq. (1.1)

$$A_0\frac{dA_0}{dx} = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d^2A_0/d\xi^2}{\xi - x}d\xi$$

A 2π-periodic solution of this equation was constructed [5]

$$A_0 = \sum_{n=-\infty}^{+\infty} b_n e^{inx};\quad b_0 = \frac{3q^2-1}{1-q^2},\quad b_n = -2q^{|n|}\ (n \neq 0),\quad 0 \le q < 1 \tag{2.1}$$

Summation of series (2.1) gives another form of the solution

$$A_0 = \frac{1+q^2}{1-q^2} - 2\frac{1-q^2}{1+q^2-2q\cos x} \tag{2.2}$$

The function $A_1(t, x)$ satisfies the linearized equation (1.1)

$$\frac{\partial A_1}{\partial t} + A_0\frac{\partial A_1}{\partial x} + \frac{dA_0}{dx}A_1 = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\partial^2 A_1/\partial\xi^2}{\xi - x}d\xi \tag{2.3}$$

which is uniform with respect to t, with coefficients that are 2π-periodic in x. We therefore seek secondary perturbations of the wave (2.2) in the class of functions

$$A_1 = \Phi_\alpha(x)\exp(i\alpha x - i\omega_\alpha t) + \text{c.c.},\quad \Phi_\alpha = \sum_{n=-\infty}^{+\infty} a_n e^{inx} \tag{2.4}$$

where $\alpha \in [0, 1)$ is is a real parameter, and ω_α and Φ_α are the complex frequency and complex eigenfunction, to be determined.

Substituting expression (2.1) and (2.4) into Eq. (2.3) we obtain a linear algebraic system for the Fourier coefficients of the perturbation

$$-\omega_\alpha a_n + (\alpha+n)\sum_{l=-\infty}^{+\infty} b_{n-l}a_l = (\alpha+n)|\alpha+n|a_n$$

We give the system the standard form of an eigenvalue problem

$$\sum_{l=-\infty}^{+\infty} A_{nl}a_l = \omega_\alpha a_n;\quad A_{nl} = (\alpha+n)\left[\left(|\alpha+n| + \frac{1+q^2}{1-q^2}\right)\delta_{nl} - 2q^{|n-l|}\right] \tag{2.5}$$

Here A_{nl} is an infinite-dimensional matrix ($-\infty < n, l < +\infty$) and δ_{nl} is the Kronecker delta.

At the initial stage of the investigation the solution of problem (2.5) was constructed using numerical methods. To do this the infinite-dimensional problem was approximated by a "truncated" finite-dimensional problem by replacing the infinite limits in the summation by finite limits ($-N < n, l < N$, $N \sim 100$); the solution of the latter was calculated using a standard subroutine from the IMSL library. An analysis of the results of the calculation enabled us to established the general form of the linearly independent solutions of problem (2.5) (we denote the number of the solution by the superscript $m \in (-\infty, +\infty)$)

$$a_n^{(m)} = c_n^{(m)}\frac{\alpha+n}{\alpha+m}q^{|n-m|} \tag{2.6}$$

and enabled us to separate the set of solutions of problem (2.5) into two classes.

The first class contains two solutions (we will give them the numbers $m = 0, -1$). All the Fourier coefficients of these solutions are non-zero, while the corresponding distributions of the values of the auxiliary coefficients $c_n^{(0)}, c_n^{(-1)}$ have the form of a "step"

$$c_n^{(0)} = \begin{cases}1, & n \ge 0\\ c^{(0)}, & n < 0\end{cases};\quad c_n^{(-1)} = \begin{cases}c^{(-1)}, & n > -1\\ 1, & n \le -1\end{cases}$$

The second class consists of an infinite set of solutions possessing the following property: the Fourier coefficients of the solutions with numbers $m \ge 1$ are equal to zero when $n < m - 1$, while the coefficients of solutions with numbers $m \le -2$ are equal to zero when $n > m + 1$. The representation of the Fourier coefficients in the form (2.6) considerably simplifies the problem of finding exact solutions of this class: a numerical solution shows that all but one non-zero auxiliary coefficients $c_n^{(m)}$ are equal to one another

$$m \ge 1: c_n^{(m)} = \begin{cases} 1, & n \ge m \\ c^{(m)}, & n = m-1; \\ 0, & n < m-1 \end{cases} \quad m \le -2: c_n^{(m)} = \begin{cases} 0, & n > m+1 \\ c^{(m)}, & n = m+1 \\ 1, & n \le m \end{cases}$$

By substituting Eqs (2.6) into Eqs (2.5) we can obtain explicit relations between the quantities $c^{(m)}$, $\omega_\alpha^{(m)}$ and the parameters q and α

$$c^{(m)} = \frac{1+(\alpha-1+m)(1-q^2)}{(\alpha-1+m)(1-q^2)^2}, \quad \omega_\alpha^{(m)} = (\alpha-1+m)^2 + \frac{1+q^2}{1-q^2}(\alpha-1+m), \quad m \ge 1$$

$$c^{(0)} = \frac{1}{1-\alpha(1-q^2)}, \quad \omega_\alpha^{(0)} = \alpha^2 - \alpha$$

$$c^{(-1)} = \frac{1}{1+(\alpha-1)(1-q^2)}, \quad \omega_\alpha^{(-1)} = -(\alpha-1)^2 - (\alpha-1)$$

$$c^{(m)} = \frac{1-(\alpha+1+m)(1-q^2)}{(\alpha+1+m)(1-q^2)^2}, \quad \omega_\alpha^{(m)} = -(\alpha+1+m)^2 + \frac{1+q^2}{1-q^2}(\alpha+1+m), \quad m \le -2$$

Hence, the fundamental periodic motion considered is neutrally stable with respect to two-dimensional secondary perturbations, since all the eigenfrequencies satisfy the equality $\operatorname{Im}\omega_\alpha^{(m)} = 0$. Summation of the corresponding Fourier series gives exact expressions for the eigenfunctions $\Phi_\alpha^{(m)}$, which we will omit for brevity.

Note that the solutions constructed possess the property of symmetry (the bar denotes complex conjugation):

$$\Phi_\alpha^{(m)} \exp(i\alpha x - i\omega_\alpha^{(m)} t) = \overline{\Phi_{1-\alpha}^{(-m-1)} \exp[i(1-\alpha)x - i\omega_{1-\alpha}^{(-m-1)} t]}$$

This property is a hidden property of the fact that the solution $\bar{A}_1$ correspond to each complex solution A_1 of Eq. (2.3): the solution with number $m < 0$ is essentially the complex conjugate of the solution with number $m' = -m - 1 \ge 0$, written in the form (2.4) (with parameter $\alpha' = 1 - \alpha$, which satisfies the inequality $0 \le \alpha' < 1$).

In the asymptotic model the flow region considered is not bounded in the longitudinal direction ($-\infty < x < +\infty$), and hence the discreteness of the spectrum has an artificial character and is generated by the above-mentioned limitation imposed on the parameter α. Really, the replacement $\alpha \to \alpha + 1$ in the solution with number $m \ge 1$ converts it into the solution with number $m + 1$, where

$$\lim_{\alpha \to 1-0} (\Phi_\alpha^{(m)} e^{i\alpha x}) = \Phi_0^{(m+1)}, \quad \lim_{\alpha \to 1-0} \omega_\alpha^{(m)} = \omega_0^{(m+1)}$$

Hence it follows that modes with numbers $m \ge 1$ describe the same secondary perturbation with continuously varying wave number $k = \alpha + m > 1$, with frequency $\omega^{(+)}$ and eigenfunction $\Phi^{(+)}$

$$A_1 = \Phi^{(+)}(x)\exp(ikx - i\omega^{(+)}t) + \text{c.c.} \tag{2.7}$$

$$\Phi^{(+)} = \frac{k-(k-1)qe^{ix}}{k(1-qe^{ix})^2} - \frac{k-(k-1)q^2}{k(1-q^2)^2} qe^{-ix}, \quad \omega^{(+)} = (k-1)^2 + \frac{1+q^2}{1-q^2}(k-1)$$

Correspondingly, the mode with number $m = 0$ is a secondary perturbation with wave number $k = \alpha < 1$

$$A_1 = \Phi^{(-)}(x)\exp(ikx - i\omega^{(-)}t) + \text{c.c.}$$

$$\Phi^{(-)} = \frac{k-(k-1)qe^{ix}}{k(1-qe^{ix})^2} - \frac{k-1-kqe^{-ix}}{k(k-1-kq^2)(1-qe^{-ix})^2} qe^{-ix}, \quad \omega^{(-)} = k^2 - k$$

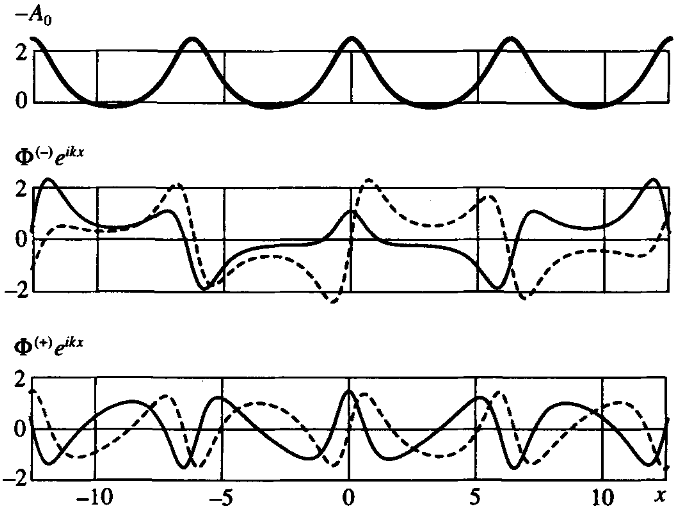

Fig. 1

When $q \to 0$ both types of perturbation acquire a sinusoidal form, and their dispersion relations in a fixed reference frame approach the form (1.2). When $k \to 1$ and q is fixed the propagation velocities of the perturbations approach a single limit, equal to the velocity of the fundamental wave; nevertheless, the corresponding forms of the oscillations do not change into one another:

$$\lim_{k \to 1-0} (\Phi^{(-)} e^{ikx}) = \frac{i}{2q}\frac{dA_0}{dx}, \quad \lim_{k \to 1+0} (\Phi^{(+)} e^{ikx}) = \frac{1}{4}\left(\frac{i}{q}\frac{dA_0}{dx} - \frac{\partial A_0}{\partial q}\right)$$

Examples of calculations of the form of the secondary perturbations of different characteristic wavelength are illustrated in Fig. 1. In the upper part we show the streamwise distribution of the displacement thickness, produced in the boundary layer by the fundamental wave when $q = 0.3$; the distributions of the complex amplitudes of the secondary perturbations of this wave for two values of the wave number $k = 0.6$ and 1.4 are shown in the middle and lower parts respectively. The real part of the complex amplitude (the continuous curve) describes the form of the perturbation at the initial instant of time, while the imaginary part (the dashed curve) is after the fourth period of the oscillations.

3. INTERACTION OF A WAVE OF SMALL AMPLITUDE WITH A SOLITON

Using the results obtained we will construct a class of exact solutions of the linearized equation (2.3), describing two-dimensional secondary perturbations of a solitary Benjamin wave. To do this we will mention a simple property of the Benjamin–Ono equation: if the function $A(t, x)$ is a solution of Eq. (1.1), the function $aA(a^2t, ax)$ will also satisfy it for an arbitrary $a > 0$. Hence, in addition to the solutions of Eq. (1.1) constructed in Section 2, there is also the solution

$$A = \frac{1}{k}A_0\left(\frac{x}{k}\right) - \frac{4\varepsilon}{\kappa(\kappa+2)k}A_1\left(\frac{t}{k^2}, \frac{x}{k}\right) + O(\varepsilon^2) \tag{3.1}$$

where the functions $A_0(x)$ and $A_1(t, x)$ are specified by relations (2.2) and (2.7), while the amplitude parameter is defined by the quantity $q = 1 - (\kappa k)^{-1}$, $\kappa > 0$.

In expansion (3.1) we will take the limit as $k \to \infty$ and change to a reference frame at rest with respect to the surface. As a result, Eq. (3.1) changes into the two-term expansion

$$A = A_0' + \varepsilon A_1' + O(\varepsilon^2)$$

which is a new solution of Eq. (1.1), written in the linear approximation. The first term in this expansion corresponds to the Benjamin soliton solution [5]; it describes a two-dimensional solitary wave of finite amplitude, propagating with a velocity κ opposite to the flow

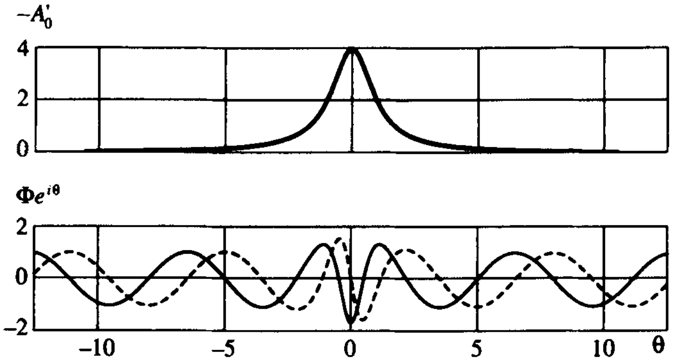

Fig. 2

$$A_0' = -\frac{4\kappa}{1+\kappa^2\theta^2}, \quad \theta = x+\kappa t \tag{3.2}$$

The second-approximation function satisfies the linear equation (2.3) (the coefficients are calculated using equality (3.2)); it corresponds to a secondary perturbation of frequency $\omega = 1$

$$A_1' = \Phi(\theta)\exp(ix-it)+\text{c.c.}, \quad \Phi = \left(\kappa\theta + i\frac{3\kappa+2}{\kappa+2}\right)\frac{\kappa\theta - i}{(\kappa\theta+i)^2} \tag{3.3}$$

The eigenfunction $\Phi(\theta)$ satisfies the condition $\Phi \to 1$ as $\theta \to \pm\infty$, and hence, upstream and downstream of the fundamental wave the perturbation has the form of a Tollmien–Schlitchting wave of the same amplitude

$$A_1' \sim \exp(ix-it)+\text{c.c.}, \quad x \to \pm\infty \tag{3.4}$$

Hence, perturbation (3.3) describes the passage of a forward Tollmien–Schlichting wave of small amplitude with frequency $\omega = 1$ through a boundary layer, perturbed by a two-dimensional solitary wave of finite amplitude. The solution corresponding to the case when $\omega \neq 1$, can be obtained from formulae (3.2)–(3.4) by making the substitution

$$t \to \omega t, \quad x \to \sqrt{\omega}x, \quad A_0' \to A_0'/\sqrt{\omega}, \quad \kappa \to \kappa/\sqrt{\omega}$$

Relation (3.4) shows that, in this model, the Tollmien–Schlichting wave, on passing through the soliton region, does not experience any resulting change of amplitude and phase.

The process described is illustrated in Fig. 2 for the case when $k = 1$ and $\omega = 1$; in the upper part we show the form of the soliton and in the lower part we show the real and imaginary components of the complex amplitude of the secondary perturbation (3.3) (the continuous and dashed curves respectively). As an analysis shows, the amplitude of the secondary perturbation is an even function of the phase variable θ and, for any κ and ω, it increases monotonically in the interval $-\infty < \theta < 0$, reaching its greatest value at the centre of the soliton:

$$|\Phi(0)| = \frac{3\kappa + 2\sqrt{\omega}}{\kappa + 2\sqrt{\omega}}$$

In the case of a short-wave perturbation ($\sqrt{\omega} \gg \kappa$) the wave experiences practically no amplification. The maximum gain is equal to 3 and occurs when the Tollmien–Schlichting wavelength is large compared with the characteristic longitudinal dimension of the soliton ($\sqrt{\omega} \ll \kappa$).

In conclusion we note that all the results obtained can also be applied to two-dimensional long-wave motions of a stratified liquid of considerable depth, since they are governed by the same equation (1.1). To describe the secondary instability it is obviously necessary to investigate the evolution of perturbations, modulated sinusoidally in a transverse direction, since, as is well known [2], the most rapidly increasing modes in the transition region have an essentially three-dimensional form. To investigate such perturbations using the asymptotic model considered here it is necessary to use a system of equations [8], which, in the case of three-dimensional perturbations, requires numerical integration with respect to the vertical variable.

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REFERENCES

1. LANDAU, L. D. and LIFSHITZ, E. M., *Fluid Dynamics*, Pergamon Press, Oxford, 1987.
2. KACHANOV, Yu. S., KOZLOV, V. V. and LEVCHENKO, V. Ya., *The Formation of Turbulence in a Boundary Layer*. Nauka, Novosibirsk, 1982.
3. Lin, C. C., On the stability of two-dimensional parallel flows. Pt 3. Stability in viscous fluid. *Quart. appl. Math.*, 1946, **3**, 4, 277–301.
4. ZHUK, V. I. and RYZHOV, O. S., Locally inviscid perturbations in a boundary layer with self-induced pressure. *Dokl. Akad. Nauk SSSR*, 1982, **263**, 1, 56–69.
5. BENJAMIN, R. B., Internal waves of permanent form in fluids of great depth. *J. Fluid Mech.*, 1967, **29**, Pt. 3, 559–592.
6. MANUILOVICH,, S. V., Spatially growing instability waves in a boundary layer at high Reynolds numbers. *Unchen. Zap. TsAGI*, 1987, **18**, 5, 35–40.
7. KACHANOV, Y. S., RYZHOV, O. S. and SMITH, F. T., Formation of solitons is transitional boundary layers: theory and experiment. *J. Fluid Mech.*, 1993, **251**, 273–297.
8. ZHUK. V. I. and RYZHOV, O. S., Three-dimensional inviscid perturbations which induce a natural pressure gradient in a boundary layer. *Dokl. Akad. Nauk SSSR*, 1989, **301**, 1, 52–56.
9. MANUILOVICH, S. V., Interaction between Tollmien–Schlichting wave and strong longitudinal irregularity of boundary-layer flow. *Book of Abstr. 4th EUROMECH Fluid Mech. Conf.,* Eindhoven, Netherlands, 2000, 254.

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